ARKIV FÖR FYSIK Band 18 nr 15

Communicated 24 February 1960 by I. Waller and H. Alfvén

Energy balance and confinement of a magnetized plasma

By B. Lehnert

With 1 figure in the text

ABSTRACT

Under laboratory conditions particle losses from a magnetized plasma will produce a neutral particle flux reentering from the surrounding vessel walls into the plasma region. Even at very high temperatures, where the plasma is almost fully ionized, this gives rise to interactions between charged and neutral particles which cannot always be neglected. Three types of interactions are discussed; ionization, charge exchange and elastic collisions with neutrals.

The conservation laws of mass, charge, momentum and energy are reconsidered for a two-fluid system with ionization, charge exchange and elastic scattering of neutral particles taken into account. It is found not to be necessary to derive the energy equation in terms of the joint action of electromagnetic and mechanical changes of state.

It has earlier been concluded from the motion of a single particle that a very strong confinement can be obtained in a rotating plasma situated in the magnetic field of a current loop (Bonnevier and Lehnert). These results are verified in a macroscopic theory including the energy balance of stationary motion. It is found that the main part of the heating of the plasma and the scattering of ions are not provided by coulomb collisions, but by collisions with reentering neutral particles. Two special cases are treated in detail, one with vanishing heat conductivity and one with infinite heat conductivity in the direction along the magnetic field. For the latter, which is more likely to occur in a high temperature plasma, the energy loss by escaping particles is deduced. The theory predicts that this loss should be several orders of magnitude less in the current loop configuration than in earlier described homopolar machines.

Finally, some comments are given on the non-existence of interchange instabilities in systems with volume currents.

I. Introduction

The ultimate goal of controlled fusion research is to produce and confine a plasma at extremely high temperatures. For this purpose the flux of high energy particles escaping from the plasma to the surrounding vessel walls should be kept at the lowest possible level. Otherwise the thermal coupling between the plasma and the walls becomes very large and, drastically speaking, the plasma does not reach the intended high temperature unless the vessel walls would be "boiled away".

As an example, there is only hope to balance the energy loss of escaping particles at some 10^8 °K by the reaction energy in a deuterium plasma with density $n = 10^{21}\,\mathrm{m}^{-3}$

if the particles are confined during times of the order of a second. Ions then have to travel about 10^6 m before they leave the confinement region. Without any confinement the energy flux to the vessel walls would become about 10^{12} watts/m^2. Suppose that the confinement is produced by a magnetic field with a mirror ratio of about 2. The velocity distribution of the particles then becomes anisotropic. Taking only coulomb collisions into account the confinement time becomes about 10^{-3} sec. and the energy loss by escaping particles about 10^9 watts/m^3 (Post 1958). At 10^6 °K the corresponding values become 10^{-5} sec. and 10^9 watts/m^3. Only if temperatures around 3×10^9 °K are reached, a mirror machine may become self-sustained (Post 1958, Garren *et al.* 1958).

Further complications in the energy balance of a thermonuclear fusion device are introduced by such effects as synchrotron radiation (Trubnikov and Kudryatsev 1958), interactions with neutral gas entering the plasma and by instabilities.

In the present paper the energy balance is studied for a system with small particle losses. An application is given by the plasma confined by the magnetic field of a current loop as suggested by Lehnert (1958*a*) and later by Greyber (1958) and Colgate and Furth (1958), heating of the plasma being provided by a transverse electric field which sets the plasma into rotation around the axis of symmetry (Lehnert 1959). In an earlier study of the motion of an individual particle it has been shown that particles under certain conditions become trapped completely inside a "bag" of magnetic field lines which is "sealed" by the centrifugal force (Bonnevier and Lehnert 1959). The situation resembles that of a planetary atmosphere, where the particles are trapped by a gravitation field, provided that their thermal velocity falls below the escape velocity. To determine the situation completely the relation between the thermal energy of the plasma and the centrifugal force has to be deduced from the energy balance. A macroscopic theory is given for this purpose in the present paper which also presents a determination of the energy losses by particles escaping from the plasma.

Some astrophysical applications to the problem of a rotating plasma are discussed more fully elsewhere (Lehnert 1960).

II. Conservation laws for the two-fluid model

Losses of charged particles from a magnetized plasma to the walls of a surrounding vessel will produce a flux of neutral particles reentering the plasma region (see also Luce 1958 and Miller 1949). Even if the plasma temperature is very high this implies that the system behaves like a partially ionized gas. Conservation of momentum for such a gas has earlier been considered by Schlüter (1950, 1951), Lucas and Schlüter (1954), Cowling (1956, 1957), Piddington (1956), Gershman and Ginzburg (1956) and Lehnert (1958 *b*). In the present paper conservation of energy will be included in the treatment.

1. *Collisions between charged and neutral particles*

In this paper the conservation theorems will be extended to include ionization processes and charge exchange collisions. Throughout the paper will be assumed that the charged particles which escape to the walls will recombine there and reenter into the gas volume in the form of neutral particles with a thermal velocity distribution nearly corresponding to the wall temperature. The possibility of using a wall material with high adsorption ability will not be discussed here.

Further, the treatment is simplified such as not to include the spontaneous emission of photons by excitation, bremsstrahlung and synchrotron radiation as well as two-stage ionization, volume recombination and multiple charge exchange collisions. The wall temperature is assumed to be considerably lower than that of the plasma. Finally, the plasma is assumed to be electrically quasi-neutral with n ions and electrons per unit volume.

Under these restrictions the reentering neutral particles may react with the plasma in three different ways. Firstly, ionization may take place with a number of ionizing collisions per unit volume and time

$$n n_n \langle \sigma_i u_i \rangle + n n_n \langle \sigma_e u_e \rangle = n \cdot \xi, \tag{1}$$

where n_n is the local neutral gas density inside the plasma, σ_i and σ_e the cross sections for ionization by means of ions and electrons, u_i and u_e the total velocities of ions and electrons, $\langle\ \rangle$ indicates effective mean values and ξ expresses the number of charged particles created per unit volume and time per charged particle being present. The second process to be taken into account is that of charge exchange between ions and neutral atoms. With the cross section given by σ_{ex} and the corresponding collision frequency by ν_{ex} the number of charge exchange collisions per unit volume and time becomes

$$n n_n \langle \sigma_{ex} u_i \rangle = n \cdot \nu_{ex} = n \cdot f \xi, \tag{2}$$

where eq. (2) is to be taken as a definition of f. Thirdly, elastic collisions with cross sections σ_{in} and σ_{en} may take place between charged and neutral particles with rates given by

$$n n_n \langle \sigma_{in} u_i \rangle = 2 n \nu_{in} = 2 n g_{in} \xi \tag{3}$$

and

$$n n_n \langle \sigma_{en} u_e \rangle = n \nu_{en} = n g_{en} \xi, \tag{4}$$

where ν_{in} and ν_{en} define the effective collision frequencies between ions and neutrals and electrons and neutrals, respectively. In average ions lose half of their momentum relative to the neutrals. The mean free path for ionization of a neutral particle which enters the plasma region with the velocity u_n becomes

$$L_i = n_n u_n / n \xi, \tag{5}$$

and the corresponding paths for exchange collisions and elastic collisions become

$$L_{ex} = L_i / f \tag{6}$$

and

$$L_{in} = L_i / g_{in}; \; L_{en} = L_i / g_{en}. \tag{7}$$

Finally, we also introduce the rate at which neutral particles are scattered elastically by ions, thereby obtaining a mean energy comparable to that of the ions and being lost to the vessel walls before undergoing any ionizing or charge exchange collision. The number of such processes per unit volume and time will be indicated by $n \cdot h \xi$ where h nearly equals $g_{in}/2$. A process of this kind may take place when L_{in} is comparable to L_i and L_{ex}. Then, the neutral particle may be "kicked" out of the plasma before it has been involved in any ionizing or charge exchange collision. This provides an additional energy loss mechanism with the same net result as that of the charge exchange. In connexion with the "DCX" experiments Barnett *et al.* (1958) have discussed the conditions for "burning out" the neutral gas inside a plasma, where neutral particles are subject to ionization and charge exchange. In cases where the mechanism just mentioned comes into operation the conditions for "burnout" may have to be reconsidered.

It should be pointed out that, before being ionized, part of the reentering neutral gas flux may become subject to elastic collisions with electrons. As a result the neutrals will diffuse through the electron gas, thereby being slowly heated and accelerated until ionization or collisions with ions take place. A detailed analysis of this situation will not be undertaken here.

Thermal diffusion effects are not included in the present treatment.

2. *Conservation of mass and charge*

With the starting points given in the preceding section the conservation laws will now be formulated for a quasi-neutral plasma. The mass density of the ion gas is given by $\varrho_i = nm_i$, the temperature by T_i, the pressure which is assumed to be isotropic by $p_i = nkT_i$ and the mass velocity by $\mathbf{v}_i$. Corresponding quantities for the electron gas are indicated by the subscript $(_e)$. Also introduce the total plasma density

$$\varrho = n(m_i + m_e) = nm = \varrho_i + \varrho_e, \tag{8}$$

the plasma pressure

$$p = p_i + p_e, \tag{9}$$

the mass velocity $\mathbf{v}$ of the plasma and the electric current density $\mathbf{i}$. The two latter quantities obey the relations

$$\mathbf{v}_i = \mathbf{v} + m_e \mathbf{i}/e\varrho; \ \mathbf{v}_e = \mathbf{v} - m_i \mathbf{i}/e\varrho. \tag{10}$$

Since charged particles are created only by ionization conservation of mass and charge is expressed by

$$\operatorname{div}\,(n\mathbf{v}_i) = \xi n - \frac{\partial n}{\partial t} = \operatorname{div}\,(n\mathbf{v}_e). \tag{11}$$

Introduce the derivative $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ and combination of eqs. (10) and (11) gives

$$\operatorname{div}\,\mathbf{v} = \xi - \frac{1}{\varrho}\cdot\frac{d\varrho}{dt} \tag{12}$$

and

$$\operatorname{div}\,\mathbf{i} = 0. \tag{13}$$

Consequently, the starting points also require that the displacement current has to be neglected and

$$\operatorname{curl}\,\mathbf{B} = \mu_0\,\mathbf{i}, \tag{14}$$

where μ_0 is the permeability in vacuum. This is a good approximation when the characteristic velocities of the phenomena in question are much less than the velocity of light. The diamagnetism of the plasma can be taken into account without introducing a modified permeability; the current density **i** in eq. (14) expresses *all* currents within the plasma explicitly (Spitzer 1956).

3. *Conservation of momentum*

Throughout this paper will be assumed that the major part of the neutral gas which interacts with the charged particles has a negligible mass velocity and a low temperature with respect to the plasma. These conditions will be discussed further in section II.5. The gain of momentum by motion in the electric and magnetic fields **E** and **B** then becomes

$$\varrho_i \frac{d_i}{dt}\mathbf{v}_i = e\,n\,(\mathbf{E}+\mathbf{v}_i\times\mathbf{B}) - \nabla p_i - \varrho_e \nu_{ei}\,(\mathbf{v}_i-\mathbf{v}_e) - \varrho_i\,(\xi+\nu_{ex}+\nu_{in})\,\mathbf{v}_i \tag{15}$$

for ions and

$$\varrho_e \frac{d_e}{dt}\mathbf{v}_e = -e\,n\,(\mathbf{E}+\mathbf{v}_e\times\mathbf{B}) - \nabla p_e + \varrho_e \nu_{ei}\,(\mathbf{v}_i-\mathbf{v}_e) - \varrho_e\,(\xi+\nu_{en})\,\mathbf{v}_e \tag{16}$$

for electrons, where ν_{ei} is the frequency of coulomb collisions and

$$d_i/dt = \partial/\partial t + \mathbf{v}_i\cdot\nabla;\quad d_e/dt = \partial/\partial t + \mathbf{v}_e\cdot\nabla.$$

By making the substitutions (8)–(10) expressions (15) and (16) can be written in terms of **v** and **i** as shown in an earlier paper (Lehnert 1958*b*). The sum of eqs. (15) and (16) gives the conservation law of the mean motion of the plasma

$$\varrho\frac{d\mathbf{v}}{dt} + \frac{m_i m_e}{e^2}(\mathbf{i}\cdot\nabla)(\mathbf{i}/\varrho) = \mathbf{i}\times\mathbf{B} - \nabla p - \alpha\varrho\mathbf{v} + \beta\mathbf{i}, \tag{17}$$

where

$$\alpha = (m\,\xi + m_i\,\nu_{ex} + m_i\,\nu_{in} + m_e\,\nu_{en})/m \tag{18}$$

and

$$\beta = m_i\,m_e(\nu_{en} - \nu_{in} - \nu_{ex})/e\,m \tag{19}$$

are "frictional coefficients" related to the loss of momentum by collision processes.

If instead eq. (15) is multiplied by $m_e/e\varrho$ and eq. (16) by $-m_i/e\varrho$ the sum of the resulting equations becomes

$$\begin{aligned}\frac{m_i m_e}{e^2\varrho}\left[\varrho\frac{d}{dt}(\mathbf{i}/\varrho) + (\mathbf{i}\cdot\nabla)\,\mathbf{v} - \frac{m_i-m_e}{e}\cdot(\mathbf{i}\cdot\nabla)\,(\mathbf{i}/\varrho)\right] = \\ = \mathbf{E}+\mathbf{v}\times\mathbf{B} - \frac{m_i-m_e}{e\varrho}\,\mathbf{i}\times\mathbf{B} + \frac{1}{e\varrho}(m_i\nabla p_e - m_e\nabla p_i) + \beta\,\mathbf{v} - \eta\,\mathbf{i},\end{aligned} \tag{20}$$

where

$$\eta = \frac{m\,m_e}{e^2\varrho}\left[\nu_{ei} + \frac{m_i}{m}\xi + \frac{m_i m_e}{m^2}(\nu_{in}+\nu_{ex}) + \left(\frac{m_i}{m}\right)^2\nu_{en}\right]. \tag{21}$$

Eq. (20) is a generalization of Ohm's law and expresses the conservation of momentum of the "slip" motion between the ions and electrons; η is a generalized electric resistivity. For further discussions of the significance of the different terms in eqs. (17) and (20) reference is made to earlier investigations (Lehnert 1958*b*).

4. *Conservation of energy*

If scalar multiplication of eq. (17) by $\mathbf{v}$ and of eq. (20) by $\mathbf{i}$ is performed the sum of the resulting equations can be written

$$\mathbf{E}\cdot\mathbf{i} - \operatorname{div}\left(p\,\mathbf{v} - \frac{m_i\,p_e - m_e\,p_i}{e\,\varrho}\,\mathbf{i}\right) = w_a + w_c + w_f. \tag{22}$$

The first term in the left hand member of eq. (22) represents the electric power input per unit volume. The second term in the same member is equal to $-\operatorname{div}(p_i\mathbf{v}_i + p_e\mathbf{v}_e)$ and represents the total work done by the pressure on a fluid element. Thus, the left hand member represents the total energy input. The right hand member tells how this energy is consumed. Here

$$\begin{aligned} w_a = \varrho_i\,\mathbf{v}_i\cdot\frac{d_i}{dt}\mathbf{v}_i + \varrho_e\,\mathbf{v}_e\cdot\frac{d_e}{dt}\mathbf{v}_e = \mathbf{v}\cdot\left[\varrho\frac{d\mathbf{v}}{dt} + \frac{m_i\,m_e}{e^2}\,(\mathbf{i}\cdot\nabla)\,(\mathbf{i}/\varrho)\right] + \\ + \frac{m_i\,m_e}{e^2\,\varrho}\,\mathbf{i}\cdot\left[\varrho\frac{d}{dt}(\mathbf{i}/\varrho) + (\mathbf{i}\cdot\nabla)\,\mathbf{v} - \frac{m_i - m_e}{e}\,(\mathbf{i}\cdot\nabla)\,(\mathbf{i}/\varrho)\right] \end{aligned} \tag{23}$$

is the total acceleration work,

$$w_c = -\,p_i\operatorname{div}\mathbf{v}_i - p_e\operatorname{div}\mathbf{v}_e = -\,p\operatorname{div}\mathbf{v} + (m_i\,p_e - m_e\,p_i)\operatorname{div}(\mathbf{i}/e\,\varrho) \tag{24}$$

the compression work and

$$w_f = w_{fi} + w_{fe} = \alpha\varrho\,\mathbf{v}^2 - 2\beta\,\mathbf{i}\cdot\mathbf{v} + \eta\,\mathbf{i}^2, \tag{25}$$

where

$$w_{fi} = \varrho_i(\xi + \nu_{ex} + \nu_{in})\,\mathbf{v}_i^2 + \varrho_e\,\nu_{ei}(\mathbf{v}_i - \mathbf{v}_e)\cdot\mathbf{v}_i \tag{26}$$

$$w_{fe} = \varrho_e(\xi + \nu_{en})\,\mathbf{v}_e^2 - \varrho_e\,\nu_{ei}(\mathbf{v}_i - \mathbf{v}_e)\cdot\mathbf{v}_e \tag{27}$$

expresses the frictional work done on ions and electrons, respectively.

It should be observed that eq. (22) can be rewritten in such a way that the pressure p is everywhere replaced by $p^* = p + B^2/2\,\mu_0$, where the second term is the "magnetic pressure". The right hand side of eq. (22) will then contain a "magnetic compression work" $-(B^2/2\,\mu_0)\operatorname{div}\mathbf{v}$, an acceleration work $-\mathbf{v}\cdot\nabla(B^2/2\,\mu_0)$ done by the magnetic pressure and an additional acceleration work $\mathbf{v}\cdot[(\mathbf{B}\cdot\nabla)\,\mathbf{B}/\mu_0]$ which is due to the fact that the magnetic field lines also may act like elastic strings. The force $(\mathbf{B}\cdot\nabla)\mathbf{B}/\mu_0$ usually does not have the character of a pressure gradient. However, this way of describing the energy balance is unnecessarily complicated and does not give any result beyond that of eq. (22), which is written in terms of the real pressures p_i and p_e and the total Lorentz force $\mathbf{i}\times\mathbf{B}$.

The obtained relation does not give any new information beyond the laws of conservation of momentum from which it is deduced. Information is still required about the coupling between thermal motion and mass motion and this can be obtained from the principle of conservation of energy (see also Lehnert 1956, 1958*c*, Pai 1957, Baños 1958 and Chu 1959).

First consider the energy balance *separately* for the ion gas inside a volume element which is fixed in space. The increase in total energy of the gas is

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2}p_i+\tfrac{1}{2}\varrho_i\,\mathbf{v}_i^2\right).$$

The energy input becomes

$$en\mathbf{E}\cdot\mathbf{v}_i-\operatorname{div}(p_i\mathbf{v}_i).$$

The plasma temperature will be assumed high enough to make the ionization work negligible as well as the neutral gas temperature.

Convection will produce a power loss

$$\operatorname{div}\left(\tfrac{3}{2}p_i\,\mathbf{v}_i+\tfrac{1}{2}\varrho_i\,\mathbf{v}_i^2\cdot\mathbf{v}_i\right)+\left(\tfrac{3}{2}p_i+\tfrac{1}{2}\varrho_i\,\mathbf{v}_i^2\right)(\nu_{ex}+h\,\xi),$$

where the first term is due to motion of the ion gas and the second to energetic neutral particles escaping from the plasma region, h being defined at the end of section II.1. Heat conduction described by the flow vector $\mathbf{Q}_i$ produces a power loss div $\mathbf{Q}_i$. Heat radiation will not be considered here. Finally, the ions will lose the power w_{fi} due to mechanical work performed against the frictional forces. The fraction δ_i of w_{fi} is thereby used to heat the electron gas and the parts of the neutral gas which do not become subject to processes defined by ν_{ex} and h. Conservation of energy requires that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{3}{2}p_i+\tfrac{1}{2}\varrho_i\,\mathbf{v}_i^2\right)&=e\,n\,\mathbf{E}\cdot\mathbf{v}_i-\operatorname{div}\,(p_i\,\mathbf{v}_i)-\operatorname{div}\,\left(\tfrac{3}{2}p_i\,\mathbf{v}_i+\tfrac{1}{2}\varrho_i\,\mathbf{v}_i^2\cdot\mathbf{v}_i\right)-\\&\quad-\left(\tfrac{3}{2}p_i+\tfrac{1}{2}\varrho_i\,\mathbf{v}_i^2\right)(\nu_{ex}+h\,\xi)-\operatorname{div}\,\mathbf{Q}_i-\delta_i\,w_{fi}.\end{aligned}\tag{28}$$

It should be stressed that $e(\mathbf{E}+\mathbf{u}_i\times\mathbf{B})\cdot\mathbf{u}_i=e\,\mathbf{E}\cdot\mathbf{u}_i$ is the total energy input from the electromagnetic field into the motion of an ion which has the total velocity $\mathbf{u}_i$. If the magnetic field $\mathbf{B}=\operatorname{curl}\,\mathbf{A}$ changes in time this will produce an energy input by means of the induced electric field part $-\partial\mathbf{A}/\partial t$. Thus, $en\mathbf{E}\cdot\mathbf{v}_i$ is the only link by which energy can be fed into the ion gas by the electromagnetic field, irrespective of the kind of electromagnetic processes being in operation (Lehnert 1956). For the electron gas the corresponding relation becomes

$$\frac{\partial}{\partial t}\left(\tfrac{3}{2}p_e+\tfrac{1}{2}\varrho_e\,\mathbf{v}_e^2\right)=-e\,n\,\mathbf{E}\cdot\mathbf{v}_e-\operatorname{div}\,(p_e\,\mathbf{v}_e)-\operatorname{div}\,\left(\tfrac{3}{2}p_e\,\mathbf{v}_e+\tfrac{1}{2}\varrho_e\,\mathbf{v}_e^2\cdot\mathbf{v}_e\right)-\operatorname{div}\,\mathbf{Q}_e-\delta_e\,w_{fe}.\tag{29}$$

Eqs. (28) and (29) express the conservation of the total energy as seen from the point of view of the charged particles. The equations express that the net change in the thermal and kinetic energy of the plasma is caused by an outflow of particles, by heat conduction, and finally by energy which is transformed into other forms than thermal and mechanical energy. In this case the latter form of energy is electromagnetic and the first term of the right hand sides in eqs. (28) and (29) gives the rate at which this energy is supplied from the electromagnetic field. "Radiation" of magneto-hydrodynamic waves is also included in eqs. (28) and (29) since these waves are characterized by the work done on the charged *particles* by the induced electric field. Consequently, it is not necessary, at least in the case of an ionized gas, to deduce the energy equation in terms of the joint action of electromagnetic and mechanical changes of state as suggested by Chu (1959).

The coupling with the electromagnetic field is now eliminated by means of the conditions for conservation of momentum. After scalar multiplication by $\mathbf{v}_i$ and $\mathbf{v}_e$ eqs. (15) and (16) are subtracted from eqs. (28) and (29) and div $\mathbf{v}_i$ and div $\mathbf{v}_e$ are substituted from eq. (11) into the obtained relations. After some deductions the result becomes

$$\begin{aligned}\varrho_i \frac{d_i}{dt}\left(\frac{3}{2}\,\frac{p_i}{\varrho_i}\right) - \frac{p_i}{\varrho_i}\cdot\frac{d_i\,\varrho_i}{dt} = &-\left(\tfrac{5}{2}\,\xi + \tfrac{3}{2}\,\nu_{ex} + \tfrac{3}{2}\,h\,\xi\right) p_i + \varrho_e\,\nu_{ei}\,(\mathbf{v}_i - \mathbf{v}_e)\cdot\mathbf{v}_i + \\ &+ \tfrac{1}{2}\,\varrho_i\,(\xi + \nu_{ex} - h\,\xi + 2\,\nu_{in})\,\mathbf{v}_i^2 - \operatorname{div}\,\mathbf{Q}_i - \delta_i\,w_{fi};\end{aligned} \tag{30}$$

$$\begin{aligned}\varrho_e \frac{d_e}{dt}\left(\frac{3}{2}\cdot\frac{p_e}{\varrho_e}\right) - \frac{p_e}{\varrho_e}\cdot\frac{d_e\,\varrho_e}{dt} = &-\frac{5}{2}\,\xi\,p_e - \varrho_e\,\nu_{ei}\,(\mathbf{v}_i - \mathbf{v}_e)\cdot\mathbf{v}_e + \\ &+ \tfrac{1}{2}\,\varrho_e\,(\xi + 2\,\nu_{en})\,\mathbf{v}_e^2 - \operatorname{div}\,\mathbf{Q}_e - \delta_e\,w_{fe}.\end{aligned} \tag{31}$$

If energy losses, heat conduction and the neutral gas inflow disappear eqs. (30) and (31) reduce to the simple adiabatic relation where the pressure is proportional to the density raised to the ratio between the specific heats. By means of the substitutions (8)–(10) the sum of eqs. (30) and (31) is transformed into the energy theorem

$$\begin{aligned}(1-\delta)\,w_f - \operatorname{div}\,\mathbf{Q} - \tfrac{3}{2}\xi\,p - \tfrac{3}{2}\,(f+h)\,\xi\,p_i = &\left[\varrho\frac{d}{dt}\left(\frac{3}{2}\,\frac{p}{\varrho}\right) - \frac{3}{2}\,\mathbf{i}\cdot\nabla\left(\frac{m_i\,p_e - m_e\,p_i}{e\,\varrho}\right)\right] - \\ &- \left[\frac{p}{\varrho}\cdot\frac{d\,\varrho}{dt} - \frac{m_i\,p_e - m_e\,p_i}{e\,\varrho^2}\,(\mathbf{i}\cdot\nabla\,\varrho) - \xi\,p - \tfrac{1}{2}\,(1+f+h)\,\xi\,\varrho_i\,\mathbf{v}_i^2 - \tfrac{1}{2}\,\xi\,\varrho_e\,\mathbf{v}_e^2\right],\end{aligned} \tag{32}$$

where w_f is the total heat power created by friction as given by eq. (25) and $\mathbf{Q} = \mathbf{Q}_i + \mathbf{Q}_e$ is the total heat flow by conduction. The left hand member is the net flow of heat, where the heat loss due to escaping charged particles is given by the third term and the heat loss due to energetic neutrals escaping from the plasma is indicated by the fourth term. The first bracket of the right hand side gives the rate of growth of the internal energy and the second bracket is the total mechanical work. The first three terms inside this bracket represent the compression work; $\xi\,p$ is the work which has to be done to "force" $n\xi$ new particles into the plasma volume at the pressure p. The last two terms within the bracket give the mechanical work done by frictional forces to accelerate $n\xi$ new particles to the full mass velocity and to balance the momentum loss of the mass motion which escapes with neutrals out of the plasma volume. The heat lost from the plasma to the neutral gas, which does not escape directly to the walls with energetic neutrals, is given by $\delta\,w_f$.

Finally, introduce the total flow of energy $\mathfrak{q}$ caused by convection and heat conduction by means of charged particles;

$$\mathfrak{q} = \tfrac{3}{2}\,(p_i\,\mathbf{v}_i + p_e\,\mathbf{v}_e) + \tfrac{1}{2}\,(\varrho_i\,v_i^2\,\mathbf{v}_i + \varrho_e\,v_e^2\,\mathbf{v}_e) + \mathbf{Q}. \tag{33}$$

The energy theorem (32) may then be rewritten by taking the sum of eqs. (28) and (29) and combine it with eqs. (22) and (11). The result becomes

$$\frac{\partial}{dt}\left(\tfrac{3}{2}p+\tfrac{1}{2}\varrho v^2+\tfrac{1}{2}\frac{m_i m_e}{e^2 \varrho}i^2\right)=w_a+w_c+(1-\delta)w_f-\operatorname{div}\mathfrak{q}-(\tfrac{3}{2}p_i+\tfrac{1}{2}\varrho_i \mathrm{v}_i^2)\cdot(f+h)\xi. \quad (34)$$

This relation describes that the growth of total energy given by the left hand member is balanced by the sum of the acceleration and compression works and the heating by friction reduced by the energy losses. The latter are produced by convection and heat conduction by charged particles together with energy losses caused by escaping energetic neutrals produced by charge exchange and by elastic scattering against ions. The present results are easily generalized to include a mass velocity of the neutral gas.

5. *Applications to gas discharge experiments*

When the theory is applied to gas discharge experiments on laboratory scale a number of approximations can be made. The following questions arise:

a. *"Creeping diffusion" and inertia forces acting on the electric current*

In the equation of motion (17) the second term of the left hand side represents the difference in acceleration of the ion and electron gases. When this effect is neglected the plasma is said to be in a state of "creeping diffusion" (Schlüter 1951, 1956). In Ohm's law (20) the left hand side represents the inertia of the charges which produce the current. A rough estimation of orders of magnitude is now made by replacing space and time derivatives by inverted characteristic lengths and times. Throughout this paper the characteristic dimensions over which all quantities vary in space are assumed to be comparable with the dimensions of the confinement region. Thus, surface currents with infinite current density and pressure jumps across discontinuity surfaces are excluded. Then, it is easily seen that the second term in the left hand member of eq. (17) as well as the left hand member of eq. (20) can be neglected, provided that the time scale of the macroscopic phenomenon to be considered is assumed to be much longer than the gyro time of an electron in the magnetic field **B** (cf. Lehnert 1958*b*).

b. *Dissipation*

A strong confining magnetic field will be assumed to exist in the systems to be treated and the dissipation of energy as well as the particle losses and the influx of neutral particles are assumed to be small. A rough estimation also shows that the terms with coefficients β and η in eqs. (17) and (20) will become small when the frequencies ξ, ν_{ex}, ν_{in} and ν_{en} all are much less than the gyro frequency of an electron.

c. *Collisions with neutral particles*

In cases where h defined in section II.1 is much less than unity a neutral particle entering the plasma region will have a good chance to be subject to ionization before it gains momentum in an elastic collision with an ion and is scattered out of the plasma. The momentum gained by elastic collisions with electrons is about m_e/m times less and can be neglected even if the velocity of the electrons is about $(m_i/m_e)^{\frac{1}{2}}$ times higher than that of the ions, provided that T_i and T_e do not differ too much. If at the same time at least one of the free paths L_i and L_{ex} defined in eqs. (5) and (6) is smaller than the thickness of the plasma region all the entering neutral gas flux will be "absorbed" in the plasma and subject to ionization or charge exchange. This implies that the energy loss defined by h and δ and the momentum loss given by ν_{in} and ν_{en} disappear in all eqs. (17), (20), (25), (32)and (34); all the losses are already represented by terms including ξ and ν_{ex}. This simplified situation will be studied henceforth.

The assumptions seem to be valid for deuterium in a temperature range from 4×10^5 to about 3×10^6 °K. According to Fite (1958) the values of σ_e vary in this range approximately from 7×10^{-21} to 3×10^{-21} m^2. Since $\sigma_{in} \approx 5.9 \times 10^{-20}$ m^2 the quantity $h \approx \frac{1}{4} \langle \sigma_{in} u_i \rangle / \langle \sigma_e u_e \rangle$ will vary between the corresponding limits 0.035 and 0.090 when $T_i \approx T_e$. Ionization by ions, which has not been taken into account, somewhat reduces these values of h. It should be observed that in this temperature range the ionization rate is mainly provided by electrons and that it exceeds the charge exchange rate. According to Little (1956) $\sigma_{ex} \leqslant 1.5 \times 10^{-20}$ m^2 for T_i in the range mentioned above giving $f \leqslant \langle \sigma_{ex} u_i \rangle / \langle \sigma_e u_e \rangle \leqslant 0.10$.

Finally, a plasma density $n \geqslant 10^{18}$ m^{-3} gives paths for ionization $L_i \leqslant 0.05$ m when $T_e = 2 \times 10^6$ °K and the neutral particles enter the plasma region with a velocity u_n corresponding to a wall temperature $T_n = 300$ °K. In such a case it seems likely that the particles will be absorbed during a single passage through the plasma.

III. Confinement and energy balance of a rotating plasma

Consider a mass of ionized gas which is rotating around an axis of symmetry. The rotation is produced by an electric field **E** the part $\mathbf{E}_\perp$ of which is perpendicular to the magnetic field **B**, as in the "Homopolar" and "Ixion" devices discussed by Baker and Anderson (1956) and Boyer *et al.* (1958). The part of **B** generated by sources out-

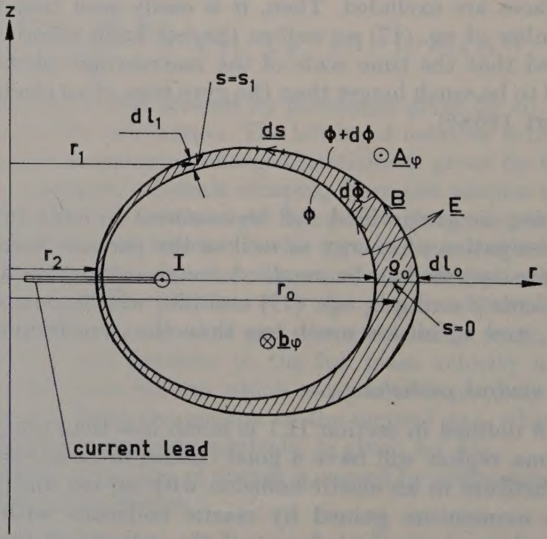

Fig. 1. The shaded area in the figure represents the cross section of a thin toroidal shell generated by the field lines of a strong poloidal magnetic field **B** which is symmetric around the z-axis of the figure. An electric field **E**, essentially at right angles to **B**, produces a rotation around the z-axis for charged particles inside the shell (cf. Bonnevier and Lehnert 1959).

side the plasma is supposed to be purely poloidal. Introduce a cylindrical coordinate system (r, φ, z) with the coordinate z along the axis of symmetry as shown in Fig. 1. Stationary, axially symmetric situations will be studied, where all macroscopic quantities are functions which do not depend explicitly on φ and t. The electric field becomes

$$\mathbf{E} = \Big/ -\frac{\partial \phi}{\partial r}, 0, -\frac{\partial \phi}{\partial z} \Big/ . \tag{35}$$

The shaded area in Fig. 1 represents the cross section of a thin toroidal shell generated by the field lines of a strong poloidal magnetic field **B**. Confinement of charged particles inside the shell will now be discussed both in terms of single particle motion and from the macroscopic point of view.

1. *Motion of an individual particle*

In a study of the forbidden regions for a single charged particle moving in the magnetic field of a current loop (cf. Fig. 1) Bonnevier and Lehnert (1959) have found that complete particle confinement can be achieved under certain conditions. This occurs when a transverse electric field is applied which produces a drift motion and a rotation around the axis of symmetry. As a consequence of the shape of the magnetic field the centrifugal work becomes very large for a particle which is "lifted" all the way from the "starting point" $(r_0, 0)$ in Fig. 1 to a point $(r_2, 0)$ in the region of the current leads which suspend the loop.

The magnetic field of Fig. 1 is now assumed to be a vacuum field and an electric field is applied across the narrow space between two electrodes which follow two toroidal surfaces generated by the field lines. **E** and $\mathbf{B} = \operatorname{curl} \mathbf{A}$ then become orthogonal all over the space between the electrodes. The shaded strip of the figure contains the magnetic flux

$$d\,\Phi = 2\pi r B\,d l = 2\pi\, d(rA_\varphi) \tag{36}$$

and the potential difference between its inner and outer surfaces is

$$d\,\Phi = -E\,d l. \tag{37}$$

The angular velocity of rotation inside the strip becomes

$$\Omega = -E/Br = 2\pi d\phi/d\Phi = d\phi/d(rA_\varphi). \tag{38}$$

Since dϕ and $d\Phi$ are constant along the strip this shows that Ω is constant along a magnetic field line. With $/u_r,\ u_\varphi + \Omega r,\ u_z/$ as the total velocity of a charged particle of mass m_ν and charge q_ν the relation

$$\begin{aligned} u_r^2 + u_z^2 = u_0^2 - (r_0^2 - r^2)\,\Omega^2 - 2\,(q_\nu/m_\nu)\,(\phi - \phi_0) + 2\,\Omega\,(q_\nu/m_\nu)\,(r\,A_\varphi - r_0\,A_{\varphi 0}) - \\ - [r_0\,u_{\varphi 0} - (q_\nu/m_\nu)\,(r\,A_\varphi - r_0\,A_{\varphi 0}) + (r_0^2 - r^2)\,\Omega]^2/r^2 \end{aligned} \tag{39}$$

has been obtained by Bonnevier and Lehnert (1959). For very strong magnetic fields the particle will be guided along a very narrow toroidal region like that indicated by the shaded area in Fig. 1. Relation (38) then shows that the third and fourth terms in the right hand member of eq. (39) cancel. Then, the particle confinement becomes complete when the centrifugal work $(m_v/2)\,(r_0^2 - r^2)\,\Omega^2$ exceeds or becomes equal to the particle energy $(m_v/2)\,u_0^2$ at the starting point $(r_0, 0)$; the point r becomes a turning point.

In the case of a plasma there are interactions between particles which have not been taken into account in the treatment of the individual particle motion. Further, the degree of confinement produced by the centrifugal field cannot be determined unless a connection between the rotation and the thermal motion is established. This requires a study of the energy balance as shown by the next paragraph.

2. *Macroscopic theory*

When the plasma confinement is studied macroscopically for configurations of the type given in Fig. 1 the assumption of isotropic pressures p_i and p_e gives a situation which is less favourable than the real one, where particles are scattered into the loss cones by collisions. Thus, the energy losses calculated in this section are to be considered as an upper limit.

The problem to be solved is that of a plasma "atmosphere" contained in a "bag" of magnetic field lines as indicated in Fig. 1 and being rotated around the axis of symmetry. Calculation will be made of the rate at which this atmosphere escapes out of the "holes" of the magnetic "bag" which are here, for the sake of simplicity, represented by the somewhat less favourable situation of a ringshaped wall situated at $r = r_1$.

a. *Basic considerations*

With the assumptions of section II.5 of a system with small losses scalar multiplication of Ohm's law (20) by **B** gives a longitudinal electric field $\mathbf{E}_{//}$, where

$$\mathbf{E}_{//}\cdot\mathbf{B} \approx -\mathbf{B}\cdot(m_i \nabla p_e - m_e \nabla p_i)/e\varrho. \tag{40}$$

Consequently, the difference in electric potential energy along a field line and along the whole configuration becomes of the order of the thermal energy of the particles. It is now supposed that a strong electric field part $\mathbf{E}_\perp$ is applied in such a way that the transverse potential difference across the configuration is much greater than that corresponding to the thermal energy of the particles. This is possible if a sufficiently strong magnetic field is applied to make the radius of gyration of a charged particle much smaller than the dimensions of the total confinement region. Thus, the situation to be studied is characterized by $E_\perp \gg E_{//}$ where the electric field is very nearly perpendicular to the magnetic field almost all over the plasma region with the exception of the Debye sheets.

Combination of eqs. (17) and (20) gives

$$\begin{aligned}\mathbf{E} + \mathbf{v}\times\mathbf{B} \approx {} & + \frac{m_i - m_e}{e}\cdot\frac{d\mathbf{v}}{dt} + (m_i \nabla p_i - m_e \nabla p_e)/e\varrho\, + \\ & + [\eta - (m_i - m_e)\,\beta/e\varrho]\,\mathbf{i} - [\beta - (m_i - m_e)\,\alpha/e]\,\mathbf{v}.\end{aligned} \tag{41}$$

A rough estimation of orders of magnitude immediately shows that the right hand member becomes much smaller than the second term of the left hand member when the gyro times of the charged particles are much shorter than the characteristic time changes of the macroscopic motion and the collision times. This is a fair approximation at small losses and strong magnetic fields.

The assumption of small energy losses also implies that the energy input $\mathbf{E}\cdot\mathbf{i}$ should be small. Since $E_\varphi = 0$ and E is relatively large the current components i_r and i_z must be small quantities, as well as the toroidal magnetic field $\mathbf{b}_\varphi$ generated by these currents. On the other hand, no restriction is made upon the induced toroidal current $\mathbf{i}_\varphi$ which gives an electromagnetic force that balances the centrifugal force and the pressure gradient in the equation of motion (17). The consequence of this is that the velocity of rotation becomes

$$v_\varphi = \Omega r \approx (\mathbf{E} \times \mathbf{B})_\varphi / \mathbf{B}^2. \tag{42}$$

Since $\mathbf{E}$ is almost perpendicular to $\mathbf{B}$ eqs. (36)–(38) show that Ω becomes nearly constant along a field line. This is consistent with the conclusion earlier drawn by Ferraro (1937) about the isorotation of a magnetized star.

Introduce the part $\mathbf{v}_{\perp rz}$ of the transverse velocity $\mathbf{v}_\perp$ which is situated in the rz-planes:

$$\mathbf{v}_{\perp rz} = \mathbf{v}_\perp - \mathbf{v}_\varphi \approx \mathbf{B} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) / B^2. \tag{43}$$

From eqs. (41) and (43) is then immediately seen that $v_{\perp rz}$ can be made very small when the magnetic field becomes strong enough. This only implies that the transverse motion of matter across the magnetic field in the rz-planes becomes very small and that nearly all the flux of charged particles out of the confinement region may take place along the magnetic field lines with the velocity $\mathbf{v}_{//}$, where $v_{//} \gg v_{rz\perp}$. Consequently, the equation (12) of continuity has the approximate form

$$\varrho \cdot \xi = \operatorname{div}(\varrho \mathbf{v}) \approx \operatorname{div}(\varrho \mathbf{v}_{//}) \approx \pm \operatorname{div}(\mathbf{B} \varrho v_{//} / B) = \pm \mathbf{B} \cdot \boldsymbol{\nabla} (\varrho v_{//} / B), \tag{44}$$

since $\operatorname{div} \mathbf{B} = 0$ and the plus sign is chosen when $\mathbf{v}_{//}$ is directed in the positive direction of $\mathbf{B}$.

The coordinate s along a magnetic field line is introduced for the upper half plane of Fig. 1 where

$$\frac{d}{ds} = \frac{B_r}{B} \cdot \frac{d}{dr}; \quad \frac{ds}{dr} = \frac{B}{B_r}. \tag{45}$$

Eq. (44) is integrated to

$$v_{//} \approx \frac{B}{\varrho} \int_0^s \frac{\varrho \xi}{B} \, ds = \frac{B}{\varrho} \int_{r_0}^r \frac{\varrho \xi}{B_r} \, dr \equiv \frac{B}{\varrho} I(s); \; (z \geqslant 0), \tag{46}$$

where $s = 0$ corresponds to $r = r_0$, which gives $v_{//} = 0$ by reasons of symmetry. Integration is made along a field line. From this expression and eq. (42) is seen that $v_{//}$ becomes much less than v_φ when the particle losses are small enough to make the necessary frequency ξ of ionization per charged particle much smaller than the angular frequency Ω. This is the situation to be studied henceforth. It implies that a charged particle travels many times around the axis of symmetry before it is lost out of the confinement volume.

If the equation of motion (17) is multiplied scalarly by **B** the result for small losses becomes

$$B\cdot\nabla p = B\frac{dp}{ds} \approx -\varrho\,\mathbf{B}\cdot[(\mathbf{v}\cdot\nabla)\,\mathbf{v}] \approx \varrho\, B_r\,\Omega^2 r, \tag{47}$$

i.e., the pressure gradient in the "atmosphere" of the magnetic field "bag" is balanced by the centrifugal field. Especially when $p_i = p_e = p/2$ the longitudinal electric field of this atmosphere can be deduced from eqs. (17) and (41) which give

$$\mathbf{E}\cdot\mathbf{B} \approx -(m_i - m_e)\, B_r\,\Omega^2 r/2e. \qquad 48)$$

Due to their smaller mass electrons have a tendency to "evaporate" more easily from the atmosphere than the ions. The electric field $\mathbf{E}_{//}$ of eq. (48) is directed in a way such as to retard electrons and accelerate ions, thereby counteracting a charge separation and producing equal escape rates of both kinds of particles. A similar result has been obtained by Spitzer (1956) for a plane atmosphere balanced by the action of gravity.

The part of the electric current which is perpendicular to **B** is obtained from eq. (17) and is given by

$$B^2\,\mathbf{i}_\perp \approx \varrho\,\mathbf{B}\times[(\mathbf{v}\cdot\nabla)\,\mathbf{v}] + \mathbf{B}\times\nabla p \approx \Big/ 0,\; -\varrho\,\Omega^2 r\, B_z\frac{\partial p}{\partial r} - B_r\frac{\partial p}{\partial z},\; 0 \Big/ \tag{49}$$

in the first approximation. Since i_r and i_z are small an estimation of orders of magnitude by means of eqs. (42) and (49) shows that i/env is of the order of the ratio between the radius of gyration of an ion and the dimensions of the confinement region, provided that thermal and rotation velocities are of the same order of magnitude. An examination of the frictional work can now be made by means of eqs. (25), (18), (19) and (21). At high temperatures and sufficiently strong magnetic fields the ratio $(m_e\,\nu_{ei}/m\alpha)(i/env)^2$ containing the coulomb collision frequency ν_{ei} becomes much less than unity. Then, the major part of the heating will be produced by collisions between charged and neutral particles, and not by coulomb collisions, i.e., the frictional work of eq. (25) is reduced to

$$w_f \approx \alpha\,\varrho\, v^2 \approx \alpha\,\varrho\,\Omega^2 r^2. \tag{50}$$

This is a situation similar to that in the solar chromosphere and in cool interstellar clouds where the dissipation is enhanced by collisions between ions and neutral particles (Piddington 1954, 1956; Cowling 1956 and Lehnert 1958*b*).

Collisions between the reentering neutral gas and the plasma will also enhance the diffusion of charged particles into the loss cones of a mirror machine and reduce the confinement times beyond the values given by pure coulomb scattering.

Finally, a few comments should also be made on the conduction of heat by charged particles. According to Kennard (1938) the heat conductivity in a gas of density n, mean thermal velocity $\bar{v}$ and a mean free path L is about $\frac{1}{2}nk\bar{v}L$. In the case of a magnetized plasma the heat conduction across a strong magnetic field will probably be reduced considerably, whereas it is unaltered by the field in the longitudinal direction (Spitzer 1956). Take an example with a hydrogen plasma with $n = 10^{22}\,\mathrm{m}^{-3}$,

$\bar{v} = 3 \times 10^7$ m/s corresponding to an electron temperature of about 10^7 °K and with $L \approx 0.3$ m. Then, the heat conductivity becomes of the order of 10^5 watts·m^{-1}. degree^{-1} along the magnetic field, which is greater than that of solid copper. Thus, at the long free paths which are actual in a discharge at very high temperatures the temperature gradient along the magnetic field lines is likely to be very small. These are about the same conditions which prevail in the exosphere, which is isothermal (Spitzer 1949, Bates 1954 and Nicolet 1954).

In the next two paragraphs examples will be given of two limiting cases, one without heat conduction and one with infinite heat conductivity along the magnetic field.

b. *Plasma without heat conduction*

First consider a simplified situation where no thermal heat flow **Q** is produced by charged particles. Further restrict the discussion not to include elastic scattering of neutrals out of the confinement volume, i.e., $g_{in} = 0$, $h = 0$ according to sections II.1 and II.5.c. The ion and electron temperatures are assumed to be approximately equal, ie., $T_i \approx T_e = T$. According to the preceding section III.2.a $\mathbf{v}_i^2 \approx \mathbf{v}_e^2 \approx \mathbf{v}^2 \approx \Omega^2 r^2$. From eqs. (43), (41) and (46) follows that a very strong magnetic field forces the convection to take place only along the field lines in the rz-plane and

$$\frac{d}{dt}(p/\varrho) = \left(v_r \frac{\partial}{\partial r} + v_z \frac{\partial}{\partial z}\right)(p/\varrho) \approx v_{//} \frac{d}{ds}(p/\varrho), \tag{51}$$

where the derivative is taken along a field line. A similar expression is obtained for $d\varrho/dt$.

Making again a rough estimation of orders of magnitude it is found from eqs. (22)–(25) and from eq. (46) that the ratio $\mathbf{E}\cdot\mathbf{i}/en\, v_{//}\, E$ becomes very small when the radius of gyration is small compared to the dimensions of the confinement region, provided that the thermal velocity of a charged particle does not exceed the angular velocity Ωr considerably. The bounding material walls at the ends of a field line are assumed to be non-conducting. Since div $\mathbf{i} = 0$ it is then reasonable to assume that the part of $\mathbf{i}$ in the rz-plane which is perpendicular to $\mathbf{E}$ also becomes small compared to $en\, v_{//}$. This simply means that the ions and electrons move with only slightly different mass velocities in the rz-plane.

With these approximations and from eq. (50) the energy theorem (32) becomes

$$(1+f)\,\xi\,\varrho\,\Omega^2 r^2 - \tfrac{5}{2}\,\xi\, p - \tfrac{3}{4} f\,\xi\, p \approx \tfrac{3}{2}\,\varrho\, v_{//} \frac{d}{ds}(p/\varrho) - \frac{p}{\varrho} v_{//} \frac{d\varrho}{ds} + \tfrac{1}{2}(1+f)\,\xi\,\varrho\,\Omega^2 r^2, \tag{52}$$

where m_i/m has been put equal to unity.

Introduce the variable $$\tau = k\,T/m\,\Omega^2. \tag{53}$$

Eqs. (47), (45) and (52) are now reduced to

$$\varrho r \frac{dr}{ds} = \frac{d}{ds}(\varrho \tau) \tag{54}$$

and $$(1+f)\,r^2-(5+\tfrac{3}{2}f)\,\tau=(v_{//}/\xi)\cdot\frac{d}{d\,s}(5\,\tau-r^2). \tag{55}$$

Two special solutions are studied. One is obtained when charge exchange is negligible ($f \ll 1$). Then eqs. (54) and (55) give the solutions

$$\tau=r^2/5 \tag{56}$$

and $$\varrho/\varrho_0=(r/r_0)^3;\ p/p_0=(r/r_0)^5. \tag{57}$$

If, instead charge exchange dominates over ionization ($f \gg 1$) the corresponding solutions become

$$\tau=2\,r^2/3 \tag{58}$$

and $$\varrho/\varrho_0=(r/r_0)^{-\frac{1}{2}};\qquad p/p_0=(r/r_0)^{\frac{3}{2}}. \tag{59}$$

The solutions (56) and (58) are plausible from the point of view of elementary considerations. An ion which is formed from a neutral particle by ionization or charge exchange will start with zero velocity and perform a cycloidal orbit in the crossed fields **E** and **B**. This corresponds to a mass motion of energy $\frac{1}{2}m_i\Omega^2r^2$ and an equal amount of "thermal" energy with two degrees of freedom. The latter is now transferred into three degrees of freedom. Further it has also to be shared with the electron which is formed when an ionizing collision is concerned, but not in the case of charge exchange. This would give $\tau=r^2/6$ and $\tau=r^2/3$ for ionizing and charge exchange collisions, respectively. However, these values of τ are too small, since a transformation of "thermal" energy from two to three degrees of freedom requires a deflection of the ion by which it will "fall" to a somewhat "lower" electric potential surface than that "grazed" by the cycloidal orbit. This somewhat increases the values of τ given by this elementary consideration, as is also shown by eqs. (56) and (58).

The results obtained so far merely serve as an illustration of the energy balance and will not be treated further. The discussion is now continued with the case of infinite heat conductivity in the direction of the magnetic field, a situation which according to section II.2.a is more likely to occur in plasma experiments at high temperatures.

c. *Plasma with infinite heat conductivity along the magnetic field*

The plasma atmosphere to be studied is now supposed to be isothermal along the magnetic field lines. This is likely to be a good approximation in a number of interesting applications. For instance, a plasma with $n=10^{22}\,\mathrm{m}^{-3}$, $T\approx10^7$ °K, $L=0.3$ m and 10 percent temperature difference over a distance of 1 m along the field lines would have a heat flow of about 10^{11} watts/m². Consequently, equilibrium requires an almost constant temperature along the magnetic field.

With the approximations of the preceding paragraph, except that of vanishing heat conductivity, eq. (47) can be integrated to

$$\varrho=\varrho_0\exp\left[-(r_0^2-r^2)/2\tau_0\right], \tag{60}$$

where τ_0 is given by the definition (53) and the constant temperature T_0 along the field line in question. Eq. (60) is analogous to the "barometer formula" where the centrifugal field has been substituted for the gravitation field. An equivalent expression has earlier been obtained by Block (1958) in connection with a study of the corotation with the sun of the interplanetary gas.

The degree of confinement depends on the magnitude of τ_0 and this can be determined from the energy balance. Since heat conduction and heat convection cannot be separated in this case the energy balance has to be considered for a complete strip like that indicated in Fig. 1, which is extended from the equatorial plane where $r = r_0$ up to a point $r = r_1$ where charged particles hit a material wall and get lost.

With the notations of Fig. 1 conservation of magnetic flux requires that

$$r\,B\,d\,l = r_0\,B_0\,d\,l_0 \tag{61}$$

and a volume element of the strip at r becomes $d\,w = 2\,\pi\,(B_0/B)\,r_0\,d\,l_0\,d\,s$. Then, the equation of continuity (12) can be integrated over the volume of the strip to give the total number of charged particles of either sign lost per unit time at r_1;

$$d\,N_1 = \frac{2\,\pi}{m}\,r_0\,B_0\,d\,l_0 \cdot I_1, \tag{62}$$

where index $(_1)$ indicates values at $r = r_1$ and use has been made of relations (46) and (61). The bounding material wall at $r = r_1$ is assumed to be nonconductive and is charged negatively in such a way that ions and electrons escape at the same rate through the thin Debye sheet close to $r = r_1$. This rate is given by eq. (62). The particles escaping at the edge of the Debye sheet correspond to an energy loss per unit time

$$d\,W_1 = \pi\,r_0\,B_0\,d\,l_0\,I_1\,\Omega^2\,(3\,\tau_0 + r_1^2). \tag{63}$$

The corresponding energy loss caused by charge exchange becomes

$$d\,W_{ex1} = \pi\,r_0\,B_0\,d\,l_0\,\Omega^2 \int_0^{s_1} f\left(\tfrac{3}{2}\tau_0 + r^2\right)\frac{\varrho\,\xi}{B}\,d\,s. \tag{64}$$

A suitable form of expressing the energy balance is now given by eq. (34) where $\mathfrak{q}$ represents the energy flow of charged particles by convection and heat conduction. For the strip in Fig. 1 the energy balance requires that

$$\int \operatorname{div} \mathfrak{q}\; d\,w = d\,W_1 = \int (w_a + w_c + w_f)\,d\,w - d\,W_{ex1}, \tag{65}$$

when elastic scattering of neutrals from the confinement region is again neglected.

With the approximations already discussed in the preceding paragraphs and with eqs. (23), (24), (42), (44), (46) and (12) the relations

$$w_a \approx \varrho\,(\mathbf{v} \cdot \boldsymbol{\nabla})\,(v^2/2) \approx \varrho\,r\,\Omega^2\,v_{\parallel}\,\frac{d\,r}{d\,s} \tag{66}$$

and

$$w_c \approx -p\,\operatorname{div} \mathbf{v} \approx -p\,\xi + \frac{p}{\varrho}\,v_{\parallel}\,\frac{d\,\varrho}{d\,s} \tag{67}$$

are obtained. Combination of these relations and eqs. (50), (46), (47) and (64) gives, after some deductions and a partial integration where it is observed that $I(0)=0$, the following result:

$$[(5+\tfrac{3}{2}f_1)\tau_0-(1+f_1)r_1^2]I_1=\int_0^{s_1} I\frac{d}{ds}[(\tfrac{3}{2}\tau_0-r^2)f]\,ds. \tag{68}$$

It should be stressed that in the deductions of eqs. (60), (68) and (56)–(59) no assumption has been made about the special geometry of the magnetic field or about the special form of the functions ξ and f. When $T=\text{const.}$ along a field line this will also be the situation for f. In the extreme case of a homogeneous magnetic field $r=r_0$ and for f being a constant the right hand sides of eqs. (55) and (68) vanish. Both equations then give the relation

$$\tau=\tau_0=r_1^2(1+f)/(5+\tfrac{3}{2}f). \tag{69}$$

The modulus of the right hand member of eq. (68) is less than $f I_1(\frac{3}{2}\tau_0+ +|r_0^2-r_1^2|)$. If the discussion is restricted to cases where ionization dominates over charge exchange ($f\ll 1$) the result becomes

$$\tau_0=r_1^2/5 \tag{70}$$

and the density and pressure distributions are

$$\varrho/\varrho_0=\exp[-5(r_0^2-r^2)/2r_1^2]=p/p_0. \tag{71}$$

In cases where r_1 is definitely less than r_0, as in the field from a current loop, the density $\varrho(r=r_1)$ at the bounding wall will be reduced very strongly. The situation described by eq. (70) gives a thermal energy which is low enough compared to the energy of the rotating mass motion to secure a good plasma confinement, i.e., a thermal velocity which is definitely below the "escape velocity" in the centrifugal field. The difference between the results (56) and (70) is caused by the high thermal conductivity along the field which reduces the temperature to a value associated with the cold material wall at $r=r_1$. Thereby it should be observed that, according to elementary kinetic theory, the heat conductivity is independent of density and is not affected by a very steep density gradient as that given by eq. (71). However, it should be kept in mind that the applicability of a fluid model becomes doubtful when the mean free path is much larger than the dimensions along the magnetic field.

An approximate expression for the energy loss by charged particles at the edge $r=r_1$ is obtained when the ions are assumed to enter the Debye sheet with a mean veolocity $\bar{u}_i=(8kT_i/\pi m)^{\frac{1}{2}}$ in the longitudinal direction. This produces a particle flux $n_1\bar{u}_i/4$ and an energy loss

$$dW_1=2\pi r_1\,dl_1\cdot n_1(\bar{u}_i/4)\cdot(\tfrac{3}{2}kT_i+\tfrac{1}{2}mv_i^2). \tag{72}$$

The normalized power loss from the strip counted per unit area of the equatiorial plane at $r=r_0$ is given by eqs. (70), (61), (72) and (71);

$$\Pi_1=\sqrt{\frac{8}{\pi}}\cdot\varrho_0(kT/m)^{\frac{3}{2}}\cdot(B_0/B_1)\exp\left[-\frac{5}{2}\left(\frac{r_0^2}{r_1^2}-1\right)\right]. \tag{73}$$

Also observe that, in addition to the reduction of density in the regions of the "holes" of a magnetic bottle, the current loop also has the advantage of a very small geometrical cross section at the edge $r = r_1$ as shown by the factor B_0 / B_1.

The results may finally be illustrated by some numerical examples. Assume a temperature of $T = 2 \times 10^6$ °K. For deuterium f and h become about 0.12 and 0.07, respectively, and eqs. (70), (71) and (73) can be used with good approximation. With a density $n = 3 \times 10^{22}\,\mathrm{m}^{-3}$ corresponding to 1 mm pressure at room temperature four examples have been calculated as shown in Table 1. The first corresponds to a homopolar machine with a mirror ratio near unity, the second to a machine with a mirror ratio of 2, the third to a current loop where r_0 is equal to twice the radius R of the loop and the fourth to a loop with an auxiliary field which "compresses" the magnetic flux into a sphere (Lehnert 1959) of radius 2.5 R. In this example r_0 has been put equal to 2.2 R. The last two examples of the table show that, as a consequence of the large ratio r_0 / r_1 and the small cross section at r_1, the energy losses by escaping charged particles can be neglected in the magnetic bottle formed by a current loop. Other losses such as those produced by radiation will probably determine the energy balance.

Table 1. The specific power loss $\prod_1$ due to particle losses from a rotating plasma confined in a magnetic field at $T = 2 \times 10^6$ $^\circ K$ and a density $n = 3 \times 10^{22}\,\mathrm{m}^{-3}$.

	r_0/r_1	B_0/B_1	$\sqrt{\frac{\pi}{8}}\left(\frac{m}{kT}\right)^{\frac{3}{2}}\Pi_1/\varrho_0$	Π_1 (watts/m²)
Homopolar machine				
(almost homogeneous field)	1	1	1	1.2×10^{11}
(mirror ratio 2)	1.414	0.500	0.041	4.9×10^9
Current loop	3.03	0.054	7.45×10^{-11}	8.9
Current loop with auxiliary field	5.50	0.115	2.30×10^{-33}	2.8×10^{-22}

According to the first two examples of the same table the energy loss in an ordinary homopolar machine is expected to be much higher. This seems to be a plauisble conclusion even if the values given in the table correspond to the isotropic pressure distribution and should be considered as an upper limit. They are also uncertain due to the fact that they have been derived from a theory for small particle losses, and this is not the situation when r_0 / r_1 obtains values close to unity.

The energy input in the homopolar device described by Anderson *et el.* (1958) is of the order of 10^{10} watts/m² at a density of about $2 \times 10^{22}\,\mathrm{m}^{-3}$ and an ion energy of about 50 eV in deuterium (Colgate 1958). With $r_0 / r_1 \approx 1$ eq. (73) gives 7×10^9 watts/m² which is of the right order, even if the applicability of eq. (73) in this case becomes somewhat doubtful. The major part of the energy input is then likely to be lost by particles moving along the magnetic field to the vessel walls. The same conclusions can also be drawn for the modified homopolar with nonuniform field (Colgate 1958) where the ratio r_0 / r_1 only exceeds unity by a small amount and where the cross section of the vessel wall at $r = r_1$ is comparatively large.

IV. The stability problem

So far no attention has been paid to the possible existence of instabilities which might increase the energy dissipation catastrophically and frustrate the attempts to obtain a high temperature. As far as can be judged at the present stage a sufficiently low plasma pressure and a sufficiently strong magnetic field is likely to give a stable configuration, at least as far as magnetohydrodynamic instabilities are concerned.

In fact, the mirror machines have been found by experiments to be stable (Post 1958). This may seem to be in contradiction with theoretical work made by a number of authors which have predicted the existence of "interchange instabilities". However, care is necessary when these results are applied to experiments. In a great number of theoretical investigations a situation has been studied where a surface current divides a vacuum region with magnetic field from a fieldfree plasma region. This is a very special problem where deformations of the bounding surface correspond to a displacement of all currents which produce the field discontinuity at the surface. A completely different situation is obtained in a plasma with a continuous pressure and current distribution. Here a displacement of a fluid element does not necessarily change the whole current distribution and the magnetic field as in the case of the sheet current. For the latter it has been found that an "interchange instability" occurs when the magnetic field lines bend concavely toward the plasma. This rule has been shown by Berkowitz *et al.* (1958) not to be applicable to mirror configurations with volume currents. These authors also point out that for a system with volume currents "instabilities can develop only because of cooperative behaviour involving an appreciable part of the system". The possibility of stabilizing the system with external magnetic fields "anchored" in solid conductors also requires further investigation.

Consequently, it is not at all sure that a system with magnetic field lines bending concavely toward the plasma should be unstable. If instabilities would in fact arise they could possibly be suppressed by using a twisted magnetic field with "shear" as suggested by Spitzer (1956). The possibility of adding a toroidal field to that generated by the current loop has earlier been discussed by Lehnert (1959). For the stationary motion of a rotating plasma described in this paper nearly all the current will flow in the φ-direction around the axis, provided that the losses are very small. The presence of an additional toroidal field $\mathbf{b}_\varphi$ hardly influences the forces and the pressure balance in such a case. If a disturbance of the axisymmetric motion arises, however, the shear produced by $\mathbf{b}_\varphi$ comes into operation.

Finally, an enhancement of the transverse diffusion rate may also be caused by asymmetric and locally varying electric fields in the plasma. Experimental results which may indicate the existence of such a phenomenon have been observed in the positive column of a discharge (Lehnert 1958*b*, *d*, Hoh and Lehnert 1959). On the other hand, no such phenomenon seems to be present in the experiments with mirror machines (Post 1958). There are a number of facts which might give possible explanation to the difference between these results, and which have to be analysed further. The former experiments are made with a relatively cold discharge of low ionization degree in a narrow tube with an electric field directed along the magnetic field lines. The latter are made with a hot, almost fully ionized gas in a mirror machine of relatively large dimensions, where almost no longitudinal electric field is present.

Royal Institute of Technology, Stockholm 70, December 19, 1959.

REFERENCES

ANDERSON, O. A., BAKER, W. R., BRATENAHL, A., FURTH, H. P., ISE, J., KUNKEL, W. B., and STONE, J. M., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *32*, 155.

Baker, W. R., and Anderson, O. A., 1956. *Project Sherwood*, Ed. by A. S. Bishop, Addison-Wesley Publ. Comp., p. 128.

Baños, A., 1958. *Electromagnetic Phenomena in Cosmical Physics*, Ed. by B. Lehnert, Cambridge Univ. Press, p. 15.

Barnett, C. F., Bell, P. R., Luce, J. S., Shipley, E. D., and Simon, A., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *31*, 298.

Bates, D. R., 1954. *The Earth as a Planet*, Ed. by G. P. Kniper, Univ. of Chicago Press, p. 635.

Berkowitz, J., Grad, H., and Rubin, H., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *31*, 186.

Block, L., 1958. Arkiv f. Fysik, *14*, 179.

Bonnevier, B., and Lehnert, B., 1959. Proc. IV. Int. Conf. on Ionization Phenomena in Gases, Uppsala, August 1959, North-Holland Publ. Company; Arkiv f. Fysik, *16*, 231.

Boyer, K., Hammel, J. E., Longmire, C. L., Nagle, D., Ribe, F. L., and Riesenfeld, W. B., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *31*, 319.

Chu, B. T., 1959. Phys. of Fluids, *2*, 473.

Colgate, S. A., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *32*, 123.

Colgate, S. A., and Furth, H. P., 1958. U.C.R.L. Report 5392, November 1958.

Cowling, T. G., 1956. M.N.R.A.S., *93*, 90.

—— 1957. *Magnetohydrodynamics*, Interscience Publ., New York and London.

Ferraro, V. C. A., 1937. M.N.R.A.S., *97*, 458.

Fite, W. L., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *32*, 405.

Garren, A., Riddell, R. J., Smith, L., Bing, G., Henrich, L. R., Northrop, T. G., and Roberts, J. E., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *31*, 65.

Gershman, B. N., and Ginzburg, V. L., 1956. Ученые Записки, *XXX*, 3.

Greyber, H. D., 1958. Aerophys. Op. Tech. Mem. No. 100, Gen. Electr., October 1958.

Hoh, F. C., and Lehnert, B., 1959. Proc. IV. Int. Conf. on Ionization Phenomena in Gases, Uppsala, August 1959, North-Holland Publ. Comp.

Kennard, E. H., 1938. *Kinetic Theory of Gases*, McGraw-Hill, New York and London.

Lehnert, B., 1956. Tellus *8*, 241.

—— 1958*a*. Nature, *181*, 331, February 1, 1958.

—— 1958*b*. Proc. of Int. School of Physics; Physics of Plasma and Astrophysical Applications, Varenna 1958; Il Nuovo Cimento, suppl. 1958.

—— 1958*c*. *Electromagnetic Phenomena in Cosmical Physics*, Ed. by B. Lehnert, Cambridge Univ. Press, p. 50.

—— 1958*d*. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *32*, 349.

—— 1959. Arkiv f. Fysik, *15*, 579; Journ. of Nuclear Energy, Part C, *1*, 40.

—— 1960. Rev. Mod. Phys.; Arkiv f. Fysik, In press.

Little, P. F., 1956. *Handbuch der Physik*, Ed. by S. Flügge, Springer-Verlag, Berlin–Göttingen–Heidelberg, p. 624.

Luce, J. S., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energe, *31*, 305.

Miller, A. R., 1949. *The Adsorption of Gases on Solids*, Cambr. Univ. Press.

Lucas, I., and Schlüter, A., 1954. Archiv der Elektrischen Übertragung, *8*, 27.

Nicolet, M., 1954. *The Earth as a Planet*, Ed. by G. P. Kniper, Univ. of Chicago Press, p. 655.

Pai, S. I., 1957. Phys. Rev., *105*, 1424.

Piddington, J. H., 1954. M.N.R.A.S. *114*, 638 and 651.

—— 1956. M.N.R.A.S., *116*, 314; *Electromagnetic Phenomena in Cosmical Physics*, Ed. by B. Lehnert, Cambridge Univ. Press, p. 141.

Post, R. F., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *32*, 245.

Schlüter, A., 1950. Z. f. Naturforschung, *5a*, 72.

—— 1951. Z. f. Naturforschung, *6a*, 73.

—— 1956. *Electromagnetic Phenomena in Cosmical Physics*, Ed. by B. Lehnert, Cambridge Univ. Press, p. 71.

Spitzer, L., 1949. *The Atmospheres of the Earth and Planets*, Ed. by G. P. Kniper, Univ. of Chicago Press, p. 213.

—— 1956. *Physics of Fully Ionized Gases*, Interscience Publ., New York and London.

Trubnikov, B. A., and Kudryatsev, V. S., 1958. Proc. II. Int. Conf. on the Peaceful Uses of Atomic Energy, *31*, 93.

Tryckt den 1 september 1960

Uppsala 1960. Almqvist & Wiksells Boktryckeri AB